WNE Linear Algebra
Final Exam
Series B

1 February 2022

Questions

Please use a single file for all questions. Give reasons to your answers or provide a counterexample. Please provide the following data in the pdf file

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem and the series.**

Each question is worth 4 marks.

Question 1.

For any matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in $M(2 \times 2; \mathbb{R})$ let

$$\operatorname{Tr}(A) = a_{11} + a_{22},$$

denote the trace of matrix A (i.e. the sum of entries of A on the main diagonal). For example, $\operatorname{Tr}\left(\left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]\right) = 1 + 4 = 5$.

Let $q\colon M(2 \times 2; \mathbb{R}) \to \mathbb{R}$ be a function given by the following formula

$$q(A) = (\operatorname{Tr} A)^2,$$

where $A \in M(2 \times 2; \mathbb{R})$.

i) explain why q is a quadratic form on the space $M(2 \times 2); \mathbb{R}$,

ii) find the matrix M of the form q relative to the basis

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

(i.e., matrix $\left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$ is identified with the vector $(a_{11}, a_{12}, a_{21}, a_{22}) \in \mathbb{R}^4$),

iii) is the form q positive definite?

Solution 1. i)

$$\left(\operatorname{Tr} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)^2 = a_{11}^2 + 2a_{11}a_{22} + a_{22}^2.$$

The function is homogeneous of degree 2.

ii)

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

iii) no, it is not,

$$\left(\operatorname{Tr}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)^2 = 0.$$

Question 2.
Let $S \in M(n \times n; \mathbb{R})$ be a matrix of an orthogonal reflection. Is

$$Q = -S,$$

a matrix of some orthogonal reflection?

Solution 2.
Yes, if S is the matrix of orthogonal reflection about subspace $V \subset \mathbb{R}^n$, then $-S$ is the matrix of orthogonal reflection about $V^\perp$.

Question 3.
Do there exist two subspaces $V, W \subset \mathbb{R}^4$ such that $\dim V = \dim W = 2$ and

$$V \cap W = \{\mathbf{0}\},$$

(i.e. the common part is the zero vector)? Give an example or explain why it is not possible.

Solution 3.
Yes, they do exist

$$V\colon \begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}, \quad W\colon \begin{cases} x_3 = 0 \\ x_4 = 0 \end{cases}.$$

Question 4.
Let $A \in M(8 \times 8; \mathbb{R})$ be a block matrix, with each block B_{ij} of size 2×2, i.e.

$$A = \left[\begin{array}{c|c|c|c} B_{11} & B_{12} & B_{13} & B_{14} \\ \hline \mathbf{0} & B_{22} & B_{23} & B_{24} \\ \hline \mathbf{0} & \mathbf{0} & B_{33} & B_{34} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & B_{44} \end{array}\right],$$

where

$$B_{11} = B_{22} = B_{33} = B_{44} = \begin{bmatrix} 2021 & 2022 \\ 2022 & 2021 \end{bmatrix}.$$

Is $\lambda = 1$ an eigenvalue of matrix A?

Solution 4.
No, it is not. For $i = 1, 2, 3, 4$

$$\det(B_{ii} - I) = \det \begin{bmatrix} 2020 & 2022 \\ 2022 & 2020 \end{bmatrix} = 2020^2 - 2022^2 \neq 0,$$

and the determinant of the upper diagonal block matrix is the product of the matrices on the diagonal, i.e.

$$w_A(1) = \det(A - I) = \det(B_{11} - I) \cdots \det(B_{44} - I) \neq 0.$$

Question 5.
For any $\alpha, \beta \in \mathbb{R}$ let $F(\alpha, \beta) \in M(3 \times 3; \mathbb{R})$ be a matrix such that

$$F(\alpha, \beta) = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}.$$

Does it follow that for any $\alpha, \beta \in \mathbb{R}$

$$(F(\alpha, \beta))^{-1} = \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{bmatrix}?$$

Solution 5.
No, it does not. It can be check by direct computation that (it is enough to check the first product only)

$$\begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\alpha & 0 \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\alpha\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$